On the Discontinuous t-norms in FRI and Linear Programming Problems

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Abstract

In this paper, we investigate the linear objective function optimization with fuzzy relational inequalities constraints where a discontinuous t-norm is considered as the fuzzy composition. Some properties satisfied in the similar problem with the continuous compositions such as max-min or max-product have been extended. The feasible solutions set is determined and its single maximum solution and minimal solutions are found. A necessary and sufficient condition is presented to realize the feasibility of the problem. Also, an algorithm has been given to optimize the linear objective function on the region comprised of the FRI constraints. Finally, we append an examples with a discontinuous t-norm Drastic Product, to illustrate.

Keywords: Linear Optimization; Fuzzy Relational Equations; Fuzzy Relational Inequalities; Fuzzy Compositions and t-norms
Introduction

In this paper, we study the linear programming problem subject to the general FRI constraints in which an arbitrary discontinuous t-norm is used to define inequalities:

\[
\min \sum_{j=1}^{n} C_j x_j \\
A o x \leq b \\
D o x \geq b \\
x \in [0,1]^n
\]  (1)

where \( I_1 = \{1, 2, \ldots, m\}, I_2 = \{m+1, m+2, \ldots, m+l\} \) and \( J = \{1, 2, \ldots, n\} \). \( A = (a_{ij})_{m \times n} \) and \( D = (d_{ij})_{l \times n} \) are fuzzy matrices such that \( 0 \leq a_{ij} \leq 1, \forall i \in I_1 \) and \( \forall j \in J \), \( 0 \leq d_{ij} \leq 1, \forall i \in I_2 \) and \( \forall j \in J \).

The vectors \( b \) and \( \tilde{b} \) are fuzzy vectors, \( c \) is the cost coefficient vector and " o " denotes the max-t composition in which \( t \) is a discontinuous t-norm. If \( a_i \) and \( d_i \) are \( i \)th row of the matrices \( A \) and \( D \), respectively, the above problem can be equivalently expressed as follows:

\[
\min \sum_{j=1}^{n} C_j x_j \\
a_i o x \leq b_i \quad i \in I_1 \\
d_i o x \geq b_i \quad i \in I_2 \\
x \in [0,1]^n
\]  (2)

where the constraints mean;

\[
a_i o x = t(a_{i1}, x_1) t(a_{i2}, x_2) \ldots \ldots t(a_{in}, x_n) \leq b_i \\
d_i o x = t(d_{i1}, x_1) \lor t(d_{i2}, x_2) \ldots \lor t(d_{in}, x_n) \geq b_i
\]  (3)

The papers of Sanchez started a development of the theory and applications of FRE treated as a formalized model for non-precise concepts [1]. The theory of FRE was originally applied in problems of the medical diagnosis [1]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [2]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering and so on. The solvability and the finding of solutions set are the primary subject concerning with FRE problems. Di Nola et al. proved that the solution set of FRE defined by continuous max-t-norm composition is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [3]. Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [4-12].

Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI) [13-18]. The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [19,20,13,21-34,14-16,18]. Fang and Li [35] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method. Wu et al. [36] improved the method used by Fang and Li, by decreasing the search domain. Chang and Shieh [19] studied the linear optimization problem constrained by fuzzy max-min relation equations and improved an upper bound on the optimal objective value. The topic of the linear optimization problem was also investigated with max-product operation [26,37,38]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [38]. Also, in [26] and [37] some necessary conditions of the feasibility and
simplification techniques were presented for solving FRE with max-product composition.

Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [39,34], max-star composition [21] and max-t-norm composition [27,30,33,22]. For example, Li and Fang [30] solved the linear optimization problem subjected to a system of sup-t equations by reducing it to a 0-1 integer optimization problem. In [27] a method was presented for solving linear optimization problems with the max-Archimedean t-norm fuzzy relation equation constraint. In [33], the authors solved the same problem with continuous Archimedean t-norm and used the covering problem rather than the branch-and-bound methods for obtaining some optimal variables. Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [39,40,15,31,42]. For example, Wu et al. [42] represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition. Dempe and Ruziyeva [43] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [39]. The linear optimization of bipolar FRE was studied by some researchers where FRE defined with max-min composition [40] and max-Lukasiewicz composition [41,31]. In [31], the authors presented an algorithm without translating the original problem into a 0-1 integer linear problem.

The optimization problem subjected to various versions of FRI could be found in the literature as well [44,13,15,26,45,18,23,24]. Yang [45] applied the pseudominimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Xiao et al. [18] introduced the latticed linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model of wireless communication emission base stations. Ghodousian and Khorram [44] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition. They used this fuzzy system to convincingly optimize the educational quality of a school (with minimum cost) to be selected by parents. In this paper, we study fuzzy relational inequalities (FRI) defined by a discontinuous t-norm, in general. At first, we determine the feasible solutions set and then we try to optimize a linear objective function subject to such these regions. The regions formulated as FRE or FRI usually introduce the vectors called maximum and minimal solutions that are used for determining the feasible regions. But, when a t-norm is discontinuous, these vectors may not be achieved by the t-norm. On the other hand, by considering the important role of these vectors in the finding of the optimal solution of a linear objective function, the optimal solution may be calculated, approximately. Nevertheless, we will show even if a t-norm is discontinuous, the optimal solution can be approximately attained with any desirable preciseness.

The remainder of the paper is organized as follows. In the section 2, we attain some results and use them to determine the feasible region of the problem (2) in the general case. In the section 3, a modification operation is given to accelerate the resolution process of the problem (2). Also, the necessary and sufficient condition is presented to realize whether the problem (2) is feasible or not. In the section 4, the problem (2) is completely resolved by the
optimization of the linear objective function considered on it. Finally, we give an example in the section 5.

The Preliminary Results

Through the paper, we consider only discontinuous triangular norms. However, when a discontinuous t-norm is used in the definition of the FRI, some component(s) of a vector \( x = [x_1, x_2, \ldots, x_n] \) may not achieved by the t-norm. For this reason, we use the notations \( x_j^- \) (if \( x_j \) is a supremum) and \( x_j^+ \) (if \( x_j \) is a infimum) that means "arbitrarily close to \( x_j \) from left, but not equal to \( x_j \) itself" and "arbitrarily close to \( x_j \) from right, but not equal to \( x_j \) itself", respectively. Also, let us call a vector \( x = [x_1, x_2, \ldots, x_n] \) pseudo-maximum, if it is the maximum solution of the solutions set and there exists at least some \( j \in J \) such that \( x_j = x_j^- \). Similarly, a vector \( x \) is said to be a pseudo-minimal, if it is a minimal solution of the solutions set and also there exists at least some \( j \in J \) such that \( x_j = x_j^+ \).

Definition (1)

Let \( t_{ij} = \{ x \in [0,1] : t(a_{ij}, x) \leq b_{ij} \} \), \( i \in I_1 \) and \( \forall j \in J \), and \( \forall j \in J \). Also, let \( S_1(a_{ij}, 1) = \{ x \in [0,1] : a_{ij} \circ x \leq 1 \} \), \( i \in I_1 \) and \( \forall j \in J \). Also, let \( S_2(d_{ij}, 2) = \{ x \in [0,1] : d_{ij} \circ x \geq 2 \} \), \( i \in I_1 \) and \( \forall j \in J \). Moreover, we set

\[
S_1(A, 1) = \{ x \in [0,1] : A \circ x \leq 1 \}
\]

and

\[
S_2(D, 2) = \{ x \in [0,1] : D \circ x \geq 2 \}
\]

Definition (2)

We introduce the notation "\( )\)" to denote both "\( )\)" and "\( ]\)" in intervals. Similarly, we use "\( (\)" instead of "\( [\)" and "\( ]\)". Also, if \( x, x \in R^n, [x, x] \) means \( {1 \cdot x_j, 2 \cdot x_j} \), \( \forall j \in J \).

Lemma (1)

\( \bar{0} \in S_1(A, 1), \text{where} \bar{0} = [0, 0, \ldots, 0] \) is the zero vector.

Proof

Since \( t(0, x) = 0, \forall x \in [0,1] \), we have \( 0 \in t_{ij} \), \( \forall i \in I_1 \) and \( \forall j \in J \). Thus, from (3), \( \bar{0} \in S_1(a_{ij}, 1) \), \( \forall i \in I_1 \), which implies \( \bar{0} \in S_1(A, 1) \).

By Lemma 1, \( S_1(A, 1) \neq \emptyset \) and \( \bar{0} \) is always the minimum solution of the set \( S_1(A, 1) \).

Lemma (2)

(a) Suppose \( \hat{x} \in t_{ij} \) for some \( i \in I_1 \) and \( j \in J \), then \( [0, \hat{x}] \subseteq t_{ij} \).

(b) Suppose \( \hat{x} \in t_{ij} \) for some \( i \in I_1 \) and \( j \in J \), then \( [1, \hat{x}] \subseteq t_{ij} \).

Proof

(a) Since \( \hat{x} \in t_{ij} \), then \( t(a_{ij}, x) \leq b_{ij} \). From the monotonicity property of t-norms, we have \( t(a_{ij}, x) \leq t(a_{ij}, \hat{x}) \), \( \forall x \in [0, \hat{x}] \) that means \( x \in t_{ij}, \forall x \in [0, \hat{x}] \). Hence, \( [0, \hat{x}] \subseteq t_{ij} \).

(b) The proof is similar to the part (a).

Lemma (3)

(a) \( S_1(a_{ij}, 1) = \prod t_{ij} \times t_{ij} \times \ldots \times t_{ij} \), \( \forall i \in I_1 \)

(b) \( S_1(A, 1) = \bigcap_{i \in I_1} t_{ij} \times \bigcap_{i \in I_1} t_{ij} \times \ldots \times \bigcap_{i \in I_1} t_{ij} \)

Proof

(a) and (b) are easily attained from (3).

Definition (3)

Let \( \bar{X} = \sup t_{ij}, \forall i \in I_1 \) and \( \forall j \in J \).

Also, for each \( i \in I_1 \), let \( \bar{X}(i) = [\bar{X}(i)_1, \bar{X}(i)_2, \ldots, \bar{X}(i)_n] \) such that:
\[
\overline{X}(i)_j = \begin{cases} 
\overline{x}_{ij}, & \overline{x}_{ij} \in t_{ij} \\
\overline{x}_{ij}, & \overline{x}_{ij} \notin t_{ij}
\end{cases}
\]

Lemma (4)

(a) For each \(i \in I_1 \) and \(j \in J\), \( t_{ij} = [0, \overline{x}_{ij} \) .

(b) \( S_1(a_i, b_i) = [0, \overline{x}_{ij} \) , \( \forall i \in I_1 \) where \( \overline{X}(i) = [\overline{X}(i), \overline{X}(i), ..., \overline{X}(i)_n ] \) such that \( \overline{X}(i)_j = \overline{x}_{ij}, \forall j \in J \) .

(c) the vector \( \overline{X}(i) \) is the single (pseudo-) maximum solution of the set \( S_1(a_i, b_i) \) , \( \forall i \in I_1 \) .

Proof

(a) Fix an \( i \in I_1 \) and \( j \in J \) . Since \( \overline{x}_{ij} = \sup t_{ij} \) then \( t(a_i, x) = [0, \overline{x}_{ij} \) when \( x > \overline{x}_{ij} \) . Hence, \( t_{ij} = [0, \overline{x}_{ij} \) . If \( \overline{x}_{ij} \in t_{ij} \) then we have also, \( [0, \overline{x}_{ij}] \subseteq t_{ij} \) from Lemma 2, part(a). Hence, \( t_{ij} = [0, \overline{x}_{ij}] \) and the proof is completed. Suppose \( \overline{x}_{ij} \notin t_{ij} \) . Therefore, \( t_{ij} \subseteq [0, \overline{x}_{ij}] \) and \( \overline{x}_{ij} \) must be a limit point. In this case, we show \( \overline{x}_{ij} = [0, \overline{x}_{ij}] \) . For this purpose, we prove \( x \in t_{ij} \) if \( x \in [0, \overline{x}_{ij}] \) . By contradiction, suppose \( x \in [0, \overline{x}_{ij}] \) and \( x \notin t_{ij} \) . Obviously, there exists some \( \varepsilon > 0 \) such that \( x < \overline{x}_{ij} - \varepsilon \) . Also, since \( \overline{x}_{ij} = \sup t_{ij} \) , there must exists some \( x_\varepsilon \in t_{ij} \) such that \( \overline{x}_{ij} - \varepsilon \leq x_\varepsilon < \overline{x}_{ij} \) . Therefore, \( x < x_\varepsilon \) that implies \( x \in [0, x_\varepsilon] \) . But, since \( x_\varepsilon \in t_{ij} \) then \( [0, x_\varepsilon] \subseteq t_{ij} \) from Lemma 2, part (a). Hence, \( x \in t_{ij} \) that is a contradiction. Thus, \( t_{ij} = [0, \overline{x}_{ij}] \) or \( t_{ij} = [0, x_\varepsilon] \) that can be briefly rewritten as \( t_{ij} = [0, \overline{x}_{ij}] \) according to our notation in Definition 2. This proves part (a).

(b) From part(a) and Lemma 3, we have \( S_1(a_i, b_i) = [0, \overline{x}_{ij}] \times [0, \overline{x}_{ij}] \times \ldots \times [0, \overline{x}_{ij}] \) . Now part (b) is obtained by the definition of \( \overline{X}(i) \) and Definition 2.

(c) At first, we note that \( \overline{X}(i) \in S_1(a_i, b_i) \) from Definition 3. Therefore, there exists at least some \( j \in J \) such that \( \overline{x}_{ij} > \overline{X}(i)_{ij} \) . From Definition 3, \( \overline{x}_{ij} > \overline{x}_{ij} \) if \( \overline{X}(i)_{ij} = \overline{x}_{ij} \) and \( \overline{x}_{ij} \geq \overline{x}_{ij} \) if \( \overline{X}(i)_{ij} = \overline{x}_{ij} \) \( \overline{x}_{ij} \notin t_{ij} \) . However, we have \( t(a_i, \overline{x}_{ij}) > b_i \) that implies \( x \notin S_1(a_i, b_i) \) from (3). \( \Box \)

Corollary 1 below is immediately attained from Lemma 4 and Definition 3.

Corollary (1)

(a) \( \overline{X}(i)_j \leq \overline{X}(i)_{ij}, \forall i \in I_1 \) and \( \forall j \in J \) .

Also, \( \overline{X}(i)_{ij} = \overline{x}_{ij} \iff \overline{x}_{ij} \in t_{ij} \),

(b) By considering \( S_1(a_i, b_i) = [0, \overline{x}_{ij}] \times [0, \overline{x}_{ij}] \times \ldots \times [0, \overline{x}_{ij}] \) \( \overline{x}_{ij} \in t_{ij} \) and \( \overline{x}_{ij} \notin t_{ij} \), \( \forall i \in I_1, \forall j \in J \) .

Lemma (5)

(a) The vector \( \overline{X} = [\overline{X}_1, \overline{X}_2, ..., \overline{X}_n] \) is the single (pseudo-) maximum solution of the set \( S_1(A, b) \) , where \( \overline{X}_{ij} = \min_{i \in I_1} \{ \overline{X}(i)_{ij} \} \).

(b) \( S_1(a_i, b_i) = [\overline{x}, \overline{x}] \) where \( \overline{x} = [\overline{x}_1, \overline{x}_2, ..., \overline{x}_n] \) such that \( \overline{X} = [\overline{X}_1, \overline{X}_2, ..., \overline{X}_n] \) and \( \overline{X} = \min_{i \in I_1} \{ \overline{X}(i)_{ij} \} = \min_{i \in I_1} \{ \overline{X}_{ij} \} \).
Proof
(a) Since \( \overline{X} \) is a maximum solution of \( \mathcal{F}(\text{FRI}) \), then \( t(a_{ij}, \overline{X}) \leq t(a_{ij}, \overline{X}(i)) \leq \overline{1} \cdot b_i \), \( \forall i \in I \), and \( \forall j \in J \). Hence, from (3), \( \overline{X} \in \mathcal{S}(a_i, b_i) \), \( \forall i \in I \), and \( \forall j \in J \). Therefore, there exists at least one \( j \in J \) such that \( x_j > \overline{X}_j \). Then, there exists at least one \( i \in I \) such that \( x_i > \overline{X}(i)_j \). If \( \overline{X}(i)_j = \overline{x}_{ij} \) (\( \overline{X}(i)_j = \overline{X}_j \)) then from Definition 3 and the proof of the part(a) of Lemma 4. Then, \( y_j \not\in \{0, \overline{x}_{ij}\} \) (\( \{x_j \not\in \{0, \overline{x}_{ij}\}\) that implies \( t(x_j, \overline{X}) > \overline{b}_i \). Therefore, \( x \not\in \mathcal{S}(a_i, b_i) \) from (3). Hence, \( x \not\in \mathcal{S}(A, b) \) that is a contradiction.

(b) The proof is easily attained from Corollary 1, part (b) and the relation

\[
\mathcal{S}(A, b) = \bigcap_{i \in I} \mathcal{S}(a_i, b_i). \quad \square
\]

Corollary (2)
According to Lemma 4 and the part(b) of Lemma 5,

\[
\mathcal{S}(A, b) = \left[0, \min_{i \in I} \left\{\overline{X}(i)\right\}\right] \times \left[0, \min_{i \in I} \left\{\overline{X}_i\right\}\right] \\ \times \cdots \left[0, \min_{i \in I} \left\{\overline{X} \right\}\right]
\]

where, \( \left[0, \min_{i \in I} \left\{\overline{X}(i)\right\}\right] \times \left[0, \min_{i \in I} \left\{\overline{X}_i\right\}\right] \times \cdots \times \left[0, \min_{i \in I} \left\{\overline{X} \right\}\right] \)

for each \( j \in J \), \( \left[0, \min_{i \in I} \left\{\overline{X}(i)\right\}\right] \times \left[0, \min_{i \in I} \left\{\overline{X}_i\right\}\right] \times \cdots \times \left[0, \min_{i \in I} \left\{\overline{X} \right\}\right] \)

if \( \min_{i \in I} \left\{\overline{X}(i)\right\} \in I_{ij} \) for some \( i' \in I \), and

\[
\left[0, \min_{i \in I} \left\{\overline{X}_i\right\}\right] \times \cdots \times \left[0, \min_{i \in I} \left\{\overline{X} \right\}\right] \]

if \( \min_{i \in I} \left\{\overline{X}(i)\right\} \not\in I_{ij} \) for all \( i' \in I \).

It can be easily shown from (4) that for each inequality \( d_i, o_x \geq 2b_i \) has the feasible solution if there exists at least some \( j \in J \) such that \( d_j \geq 2b_j \). Also, if this condition is satisfied then \( \overline{1} = [1, 1, \ldots, 1] \) is the single maximum solution of the inequality \( d_i, o_x \geq 2b_i \). These results can be abbreviated in the following corollary.

Corollary (3)
(a) \( \mathcal{S}(A, b) \neq \emptyset \) if \( d_j \geq 2b_j, \forall i \in I_2 \) and \( \forall j \in J \).

(b) For each \( i \in I_2, \mathcal{S}_2(d_i, b_j) \neq \emptyset \) if \( \exists j \in J \) such that \( 2b_j \neq \emptyset \).

(c) \( \mathcal{S}_2(D, b) \neq \emptyset \) if \( \forall i \in I_2, \exists j \in J \) such that \( 2b_j \neq \emptyset \).

Definition (4)
For each \( i \in I \), we set

\[
x_{ij} = \begin{cases} \inf \{2t_{ij} \} & d_j \geq 2b_j \\ \infty & d_j < 2b_j \end{cases}
\]

Definition (5)
Let \( J_1 = \{ j \in J : d_j \geq 2b_j \} \). For each \( i \in I_2 \) and \( j \in J_1 \), let \( T_{ij} = A_j \times A_{j+1} \times \cdots \times A_n \) in which

\[
A_k = \begin{cases} 2t_{ij} & k = j \\ [0, 1] & k \neq j \end{cases}
\]

Lemma (6)
(a) \( \mathcal{S}_2(d_i, b_j) = \bigcup_{j \in I_1} T_{ij} \)

(b) \( \mathcal{S}_2(D, b) = \bigcap_{i, j \in I_1} T_{ij} \)

Proof
The proof is attained from (4) and Definitions 1 and 5. \( \square \)

**Definition (6)**
For each \( i \in I_2 \) and \( j \in J_i \), we define \( \bar{X}(j)^+ = [\bar{X}(j)_1^+, \bar{X}(j)_2^+, \ldots, \bar{X}(j)_n^+] \) where

\[
\bar{X}(j)^+_k = \begin{cases} x_{ij} & k = j \text{ and } x_{ij} \in 2t_j \\ x_{ij} & k = j \text{ and } x_{ij} \notin 2t_j \\ 0 & k \neq j \end{cases}
\]

From (4) and Definitions 4 and 6, \( \bar{X}(j)^+ \) is the feasible (pseudo-)minimal point in the set \( S_2(d_i, 2b_i), \forall i \in I_2 \) and \( j \in J_i \). Also, we have the lemma below proved with a proof similar to that of Lemma 4.

**Lemma (7)**

(a) For each \( i \in I_2 \) and \( j \in J_i, \bar{t}_j^+ = \langle x_{ij}, 1 \rangle \).

(b) If \( S_2(d_i, 2b_i) \neq \emptyset \),
\[
S_2(d_i, 2b_i) = \bigcup_{j \in J_i} \langle \bar{X}(j), 1 \rangle, \forall i \in I_2,
\]
where \( \bar{X}(j) = [\bar{X}(j)_1, \bar{X}(j)_2, \ldots, \bar{X}(j)_n] \) such that \( \bar{X}(j)_k = \begin{cases} x_{ij} & k = j \\ 0 & k \neq j \end{cases} \)

(c) The vectors \( \bar{X}(j)^+ \) are feasible (pseudo-)minimal solutions of the set \( S_2(d_i, 2b_i), \forall i \in I_2 \) and \( \forall j \in J_i \).

The following corollary is equivalent to Corollary 1 and it is attained from Definition 6 and Lemma 7.

**Corollary (4)**

(a) \( \bar{X}(j)^+ \subseteq \bar{X}(j)^+, \forall i \in I_2 \) and \( \forall j \in J_i \). Also, \( \bar{X}(j)^+ = \bar{X}(j)^+ \) iff \( x_{ij} \in 2t_j \).

(b) By considering \( \bar{t}_j = [0, 1] \times \ldots \times [x_{ij}, 1] \times \ldots \times [0, 1] \), where \( \langle x_{ij}, 1 \rangle \) is at the \( j \)th position, \( \langle x_{ij}, 1 \rangle = [x_{ij}, 1] \) if \( x_{ij} \in 2t_j \) and

\[
\langle x_{ij}, 1 \rangle = (x_{ij}, 1) \text{ if } x_{ij} \in 2t_j, \forall i \in I_2 \text{ and } \forall j \in J_i.
\]

**Definition (7)**
Let \( e = (e(m+1), e(m+2), \ldots, e(m+1)) \) such that \( e(i) = j \in J_i, \forall i \in I_2 \) and \( E \) is the set of all the vectors \( e \).

**Definition (8)**
For each \( e \in E \), we define \( I_1(e) = \{ i \in I_2 : e(i) = j \} \).

Also we introduce \( I_{\min}(j) = \{ i \in I_2 : t_j \neq \emptyset \} \) and \( I_{\min}(j) = \{ i \in I_2 : t_j \neq \emptyset \} \).

**Lemma (8)**

(a) \( S_2(D, 2b) = \bigcap_{e \in E \cap I_2} \bar{T}_{I_{\min}(j)} \).

(b) \( \bar{T}_{I_{\min}(j)} = B_1 \times B_2 \times \ldots \times B_n \) such that

\[
B_j = \begin{cases} [0, 1] & I_j(e) = \emptyset \\ [x_{ij}^*, 1] & I_j(e) \neq \emptyset \text{ and } x_{ij} > x_{ij}^* \\ [x_{ij}^*, 1] & I_j(e) \neq \emptyset \text{ and } x_{ij} < x_{ij}^* 
\end{cases}
\]

Where

\( e \in E, x_{ij}^* = \max_{i \in I_{\max}(j) \cap I_{\min}(j)} \{ x_{ij} \} \) and \( x_{ij} = \max_{i \in I_{\min}(j)} \{ x_{ij} \} \)

**Proof**

(a) The result is proved from Definition 1 and 7, the part (b) of Lemma 6.

(b) The proof is attained from Definition 5 and 8, the part (a) of Lemma 7 and this fact that \( I_j(e) \neq \emptyset \) then

\[
B_j = \left( \bigcap_{i \in I_{\max}(j)} (x_{ij}, 1) \right) \bigcap \left( \bigcap_{i \in I_{\min}(j)} [x_{ij}, 1] \right) = \left( \bigcap_{i \in I_{\min}(j)} (x_{ij}, 1) \right) \bigcap \left( \bigcap_{i \in I_{\min}(j)} [x_{ij}, 1] \right).
\]
Definition (9)

For each \(e \in E\), we define
\[X(e)^+ = \max \{X(e)^+\} \text{ and } X(e) = \max \{X(e)^+\}\]

Lemma (9)

\[(a) \quad S_2(D, 2b) = \bigcup_{e \in E} \langle X(e) \rangle. \]

\[(b) \quad \text{Each (pseudo) minimal solution of the set } S_2(D, 2b) \text{ belongs to the set } \{X(e)^+\} = \bigcup_{e \in E} \langle X(e) \rangle. \]

Proof (a) From Lemma 8, we have
\[\bigcap_{i \in I} T_{t(i)} = B_1 \times B_2 \times \ldots \times B_n. \]
Let \(x_{ij} = 0\) if \(i \in I^1\), and \(x_{ij} = \max \{x_{ij}^*, x_{ij}^0\}\) if \(i \in I_0\), where \(x_{ij}^*\) and \(x_{ij}^0\) have been defined in Lemma 8. By Lemma 8, \(B_j = [0, 1] = [x_{ij}, 1]\) if \(i \in I_0\) and \(B_j = [x_{ij}, 1]\) if \(i \in I^1\). Therefore, \(B_j = [x_{ij}, 1] = \langle X(e) \rangle\) from Definition 9 and the part (b) of Lemma 7. Thus,
\[\bigcap_{i \in I^1} T_{t(i)} = \langle X(e) \rangle = \bigcup_{e \in E} \langle X(e) \rangle. \]
This implies \(S_2(D, 2b) = \bigcup_{e \in E} \langle X(e) \rangle\)

(b) The proof is attained from the part (a) and Definition 9. \(\square\)

Remark (1)

From corollary 3, \(S_2(d, 2b) \neq \emptyset\) iff \(\hat{1} \in S_2(d, 2b)\). Also, if there exists some \(i \in I_1\) and \(j \in J_1\) such that \(x_{ij} = 0\), we attain \(S_2(d, 2b) = [0, 1]\). Therefore, in this case, we can remove the \(i\)th row of the matrix \(D\) without any effect on the solutions sets.

Resolution of the Solutions Set and Modification

Theorem (1)

The following statements are equivalent:

\[(a) \quad S(A, D, 1b, 2b) \neq \emptyset\]

\[(b) \quad X^- \in S_2(D, 2b). \]

\[(c) \quad \text{there exists at least some } e \in E \text{ such that } X(e)^+ \in S_2(A, 1b)\]

\[(d) \quad \text{there exists at least some } e \in E \text{ such that } \langle X(e), X^- \rangle \neq \emptyset. \]

Proof (a) implies (b). Since \(S_2(A, D, 1b, 2b) \neq \emptyset\) from Lemma 5 (part (b)) and Lemma 9 (part (a)), there exists some \(e \in E\) such that \([0, X] \cap \langle X(e), \hat{1} \rangle \neq \emptyset\). Hence, \(X^- \in \langle X(e), \hat{1} \rangle\) that requires \(X^- \in S_2(D, 2b)\) from Lemma 9 (part (a)). (b) implies (c). If \(X^- \in S_2(D, 2b)\) then \(X^- \in \langle X(e), \hat{1} \rangle\) for some \(e \in E\), from Lemma 9 (part (a)). For this \(e\), we have \(X(e)^+ \leq X^-\) that requires \(X(e)^+ \in [0, X^-]\). Therefore, \(X(e) \in S_2(A, 1b)\) from Lemma 5 (part (b)).

(c) implies (d). We show that \(X(e)^+ \in S_2(A, 1b)\) implies \(\langle X(e), X^- \rangle \neq \emptyset\). By contradiction, suppose \(X(e)^+ \in S_2(A, 1b)\) for some \(e \in E\). Since \(\langle X(e), X^- \rangle = \langle X(e), X \rangle \times \langle X(e), X \rangle \times \ldots \times \langle X(e), X \rangle\), then there exists at least some \(j \in J\) such that \(\langle X(e), X \rangle = \emptyset\). The interval \(\langle X(e), X \rangle\) may take the form of any intervals \([X(e), X], [X(e), X], [X(e), X], [\ldots, [X(e), X]]\) or \([X(e), X]\).

Consider \(\langle X(e), X \rangle = [X(e), X] = \emptyset\) (the proof is similar for other cases). Therefore, since \(\langle X(e), X \rangle = [X(e), X] = \emptyset\) then \(X(e) > X^-\)
that implies $\bar{X}(e)^{+}_j > \bar{X}_j$ from Corollary 4 (part(a)). Hence, $\bar{X}(e)^{+}_j \notin [0, \bar{X})$. But, this contradicts from $\bar{X}(e)^{+}_j \in S(A, b)$. Lemma 5 (part(b)). (d) implies (a). Suppose $\langle \bar{X}(e), \bar{X} \rangle \neq \emptyset$ for some $e \in E$. We consider $x \in \langle \bar{X}(e), \bar{X} \rangle$. Therefore, $x \in \langle \bar{X}(e), \bar{I} \rangle$ and $x \in [0, \bar{X})$. These facts require $x \in S_2(D, b)$ from Lemma 9 (part(a)) and $x \in S_1(A, b)$ from Lemma 5 (part(b)). Hence, $x \in S_1(A, b) \cap S_2(D, b) = S(A, b)$ that yields the proof. □

**Theorem (2)**

If $S(A, D, b, 2b) \neq \emptyset$, then $S(A, D, b, 2b) = \bigcup_{e \in E} \langle \bar{X}(e), \bar{X} \rangle$, where $E' = \{ e \in E : \langle \bar{X}(e), \bar{X} \rangle \neq \emptyset \}$.

**Proof**

The proof is attained from Lemmas 5, 9, Theorem 1 and the quality $S(A, D, b, 2b) = S_1(A, b) \cap S_2(D, b)$. □

**Corollary (5)**

(a) $\bar{X}$ is the single (pseudo-)maximum solution of the set $S(A, D, b, 2b)$.

(b) Each (pseudo-)minimum solution of the set $S(A, D, b, 2b)$ belongs to the set 

$\{X(e)^{+} : e \in E' \}$. Also, $X(e)^{+} \leq \bar{X}$ iff $e \in E'$.

From Theorem 2, if $S(A, D, b, 2b) \neq \emptyset$ then it is determined by single (pseudo-)maximum solution and finite (pseudo-)minimal solutions it is sufficient to consider only vectors $e \in E'$. Actually, some selection of the vectors $e$ leads to infeasible vectors $X(e)^{+}$. In an efficient method, It is better to realize such these vectors $e \in E$ and remove them before generating the solutions $X(e)^{+}$. Such these vectors are distinguished by property stated in the following lemma.

**Lemma (10)**

$e \in E'$ iff the following two conditions hold:

(a) $x_{ie(i)} \leq \bar{X} - \bar{X}_j$, for each $i \in I_j$ such that $x_{ie(i)} \in \bar{X} - \bar{X}_j$.

(b) $x_{ie(i)} \leq \bar{X} - \bar{X}_j$, for each $i \in I_j$ such that $x_{ie(i)} \notin \bar{X} - \bar{X}_j$.

**Proof**

Suppose $e \in E'$ and $e(i') = j'$ for a fixed arbitrary $i' \in I_j$. From Corollary 5 (part(b)), $X(e)^{+} \leq \bar{X}$ that implies $X(e)^{+}_j \leq \bar{X}_j$, $\forall j \in J$. Especially, we have $X(e)^{+}_j \leq \bar{X}_j$. On the other hand, from Lemma 8, we have

$X(e)^{+}_j = \max \{ \max_{i \in I_j(e)} \{ x_{ij} \}, \max_{i \in I_{ie}(j)} \{ x_{ij} \} \}$

From (\ast) and the assumption $i \in I_j(e)$, we have $x_{ij} \leq X(e)^{+}_j$. Then $x_{ij} \leq \bar{X}_j$. But, since $e(i) = j', \forall i \in I_j(e)$, and $i' \in I_j(e)$ then $e(i') = j'$ that converts the inequality $x_{ij} \leq \bar{X}_j$ into $x_{ie(i')} \leq \bar{X}_j$. Since, the last inequality is satisfied for each arbitrary $i' \in I_j$ then the necessary is proved. Conversely, suppose the conditions (a) and (b) are satisfied. If $I_j = \emptyset$ then $X(e)^{+}_j = 0 \leq \bar{X}_j$. Otherwise, by considering the conditions (a) and (b), we have $X(e)^{+}_j = \max \{ \max_{i \in I_j(e)} \{ x_{ij} \}, \max_{i \in I_{ie}(j)} \{ x_{ij} \} \}$, which implies $e \in E'$ from Corollary 5 (part(b)). □

**Corollary (6)**

In order to accelerate the finding process of the solutions $X(e)^{+}$, we can reduce the set $J_i$...
by removing each \( j \in J \) such that \( x_{ij} > \overline{X}_j \) (if \( x_{ij} \in \mathbb{Z}t_{ij} \)) or \( x_{ij} > \overline{X}_j \) (if \( x_{ij} \notin \mathbb{Z}t_{ij} \)), \( \forall i \in I_2 \), before selecting the vectors \( e \) to construct such these solutions \( X(e)^+ \).

**Definition (10)**

Let \( M = (m_{ij})_{i,n} \) be matrix whose components are defined as following for each \( i \in I_2 \) and each \( j \in J \):

\[
m_{ij} = \begin{cases} 
  x_{ij} & d_{ij} \geq 2b_i \text{ and } x_{ij} \in \mathbb{Z}t_{ij} \\
  \infty & d_{ij} < 2b_i 
\end{cases}
\]

We construct the modified matrix \( M^* = (m_{ij}^*)_{i,n} \) from the matrix \( M \) as follows:

\[
m_{ij}^* = \begin{cases} 
  \infty & d_{ij} \geq 2b_i \text{ and } m_{ij} \geq \overline{X}_j \\
  m_{ij} & \text{otherwise}
\end{cases}
\]

**Corollary (7)**

According to the definition of the matrix \( M^* \), if we set \( J_1^* = \{ j \in J : m_{ij}^* \neq \infty \} \) and \( E^* = \{ e \in E : e(i) \in J_1^*, \forall i \in I_2 \} \) then we have \( J_1^* \subseteq J_1 \) and \( E^* = E^* \subseteq E \).

**Theorem (3)**

(a) The \( i \) 'th row of the matrix \( D \) can be removed without any effect on the solutions set, if \( m_{ij} = 0 \) for at least some \( j \in J \).

(b) \( S(A,D,1b,2b) = \emptyset \) if there exists at least some \( i \in I_2 \) such that \( m_{ij} = \infty, \forall j \in J \).

(c) \( S(A,D,1b,2b) \neq \emptyset \) iff there exists at least some \( j \in J \) such that \( m_{ij} = \infty, \forall i \in I_2 \).

**Proof**

(a) From Definition10, if \( m_{ij} = 0 \) for some \( i \in I_2 \) and some \( j \in J \), then \( x_{ij} = 0 \). Now, the proof is attained from the remark1.

(b) If there exists some \( i \in I_2 \) such that \( m_{ij} = \infty, \forall j \in J \), then we have \( d_{ij} < 2b, \forall j \in J \), from Definition10. Then, the result is proved from Corollary3.

(c) Suppose \( S(A,D,1b,2b) \neq \emptyset \). By contradiction, suppose \( i' \in I_2 \) such that \( m_{ij}^* = \infty, \forall j \in J \).

From part(a) and the assumption \( S(A,D,1b,2b) \neq \emptyset \), there must exist at least some \( j \in J \) such that \( m_{ij}^* = \infty \). Therefore, since \( m_{ij}^* = \infty, \forall j \in J \), then for each \( j \in J_1^* \), \( x_{ij}(or \overline{x}_{ij}) > \overline{X}_j \).

Hence, we have:

\[
X(e^*)_{e(i)} \geq x_{ij}(or \overline{x}_{ij}) > \overline{X}_j,
\]

Hence, \( X(e^*) \neq \overline{X} \), \( \forall e \in E \) that implies \( e \in E' \) from Corollary5 (part(b)). Thus, \( E' = \emptyset \) and then \( S(A,D,1b,2b) \neq \emptyset \) from Theorem2 that is a contradiction. Conversely, suppose, \( \forall i \in I_2 \), \( \exists j_i \in J \) such that \( m_{ij}^* = \infty \). We select the vector \( e \in E \) such that \( e(i) = j_i, \forall i \in I_2 \). Since, \( m_{ij}^* = \infty, \forall i \in I_2 \) then \( x_{ij}(or \overline{x}_{ij}) \leq \overline{X}_j, \forall i \in I_2 \).

Therefore, \( X(e)^+ = \max \{ X(e)^+ \} \leq \overline{X} \) that implies \( e \in E' \), from Corollary5 (part(b)). Thus, \( \langle X(e), \overline{X} \rangle \neq \emptyset \) and then \( S(A,D,1b,2b) \neq \emptyset \) by Theorem2.

**Theorem (4)**

Suppose \( S(A,D,1b,2b) \neq \emptyset \) and \( e \in E' \).

\( e \in E' \) iff \( X(e)^+ \in S(A,D,1b,2b) \).

**Proof**

Suppose \( X(e)^+ \in S(A,D,1b,2b) \) then \( X(e)^+ \leq \overline{X} \).
and then \( e \in E' \) by corollary 5 (part(b)). Now, the result is proved from Corollary 7. Conversely, suppose \( e' \in E^* \). Therefore, \( x_{e'(i)} \in i \in I \) (or \( x^+_{e'(i)} \leq X^-_{e'(i)}, \forall i \in I_2 \) and then \( iX(e'(i))^+ \leq X^-, \forall i \in I_2 \). Hence, \( X(e')^+ = \max_{i \in I_2} \{iX(e'(i))^+\} \leq X^- \) that implies \( X(e') \in [0, X] \) and proves the result from Lemma 5 (part(b)). □

**Linear Objective Function Optimization**

According to the scheme used in the literature [13,15,30], we convert the problem (2) into the following two sub-problems:

\[
\min \sum_{j=1}^{n} c_j^+ x_j \quad \min \sum_{j=1}^{n} c_j^- x_j
\]

\[
\begin{align*}
A o x & \leq 1b \quad (*) & A o x & \leq 1b \quad (*2) \\
D o x & \geq 2b & D o x & \geq 2b \\
x & \in [0,1]^n & x & \in [0,1]^n
\end{align*}
\]

Where \( c_j^+ = \max \{c_j, 0\} \) and \( c_j^- = \min \{c_j, 0\} \) for \( j = 1, 2, \ldots, n \).

It is easy to prove that \( X^- \) is the optimal solution of (*2). Also, the optimal solution of (*1) is \( X(e) \) for some \( e \in E' \).

**Theorem (5)**

If \( X(e*) \) is the optimal solution of (*1) then \( x^* = x_1^*, x_2^*, \ldots, x_n^* \) defined below is the optimal solution of the problem(2).

\[
x_j^* = \begin{cases} 
X^- & c_j < 0 \\
X(e*) & c_j \geq 0
\end{cases} \quad j = 1, 2, \ldots, n
\]

**Proof**

See the proof of Corollary 4.1 in [11].

Algorithm 1 below abbreviates the solutions steps of the problem (2).

**Algorithm (1):** Given the problem (2).

1. Calculate \( X(i)^- \), \( \forall i \in I_1 \) by Definition 3.
2. Calculate \( X = \min_{i \in I_2} \{X(i)^-\} \).
3. Calculate the matrices \( M \) and \( M^* \) by Definition 10.
4. If there exists at least some \( i \in I_2 \) such that \( m^*_i = \infty, \forall j \in J \) then stop. Problem is infeasible.
5. Select the vectors \( e \in E^* \) and calculate the vectors \( X(e) \) from the matrix \( M^* \).
6. Find the optimal solution of \( (*) \) between the vectors \( X(e) \) attained in step 5.
7. Calculate the optimal solution of the problem (2) by Theorem 5.

**Numerical Examples**

**Example (1)**

Consider the problem below with Drastic product t-norm \( \min c'x = -0.5x_1 + 3x_2 + x_3 - x_4 \)

\[
\begin{bmatrix}
0.89 & 0.72 & 0.5 & 1 \\
0.62 & 0.42 & 0.7 & 0.7 \\
1 & 0.1 & 1 & 0.58 \\
0.43 & 0.94 & 0.12 & 1
\end{bmatrix}
\]

\[
\begin{align*}
& o x \leq [0.8 \ 0.63 \ 0.58 \ 0.43] \\
& o x \leq [0.39 \ 0.44] x \in [0,1]^n
\end{align*}
\]

Since Drastic product t-norm is a right-continuous t-norm then we must have minimal solutions instead of pseudo-minimal ones. However, the maximum solution is found after calculating, In this example. We have...
\[ t_{11} = [0.1] \quad t_{12} = [0.1] \quad t_{13} = [0.1] \quad t_{14} = [0.08] \]
\[ t_{21} = [0.1] \quad t_{22} = [0.1] \quad t_{23} = [0.1] \quad t_{24} = [0.1] \]
\[ t_{31} = [0.058] \quad t_{32} = [0.1] \quad t_{33} = [0.058] \quad t_{34} = [0.1] \]
\[ t_{41} = [0.1] \quad t_{42} = [0.1] \quad t_{43} = [0.1] \quad t_{44} = [0.043] \]

\[ \widetilde{X}(1) = [1^{*} \ 1 \ 1 \ 0.8] \quad \widetilde{X}(3) = [0.58 \ 1 \ 0.58 \ 1] \]
\[ \widetilde{X}(2) = [1 \ 1 \ 1^{*} \ 1] \quad \widetilde{X}(4) = [1 \ 1 \ 1^{*} \ 1] \]

So, the pseudo-maximum solution is \[ \widetilde{X} = [0.58 \ 1 \ 0.58 \ 0.43] \]. Also, we have
\[ t_{11}^{*} = \{1\} \quad t_{12} = \emptyset \quad t_{13} = [0.39, 1] \quad t_{14} = [0.39, 1] \]
\[ t_{21} = [0.44, 1] \quad t_{22} = \{1\} \quad t_{23} = \emptyset \quad t_{24} = [0.44, 1] \]

Therefore,
\[ M = \begin{bmatrix} 1 & \infty & 0.39 & 0.39 \\ 0.44 & 1 & \infty & 0.44 \end{bmatrix} \quad M^* = \begin{bmatrix} \infty & \infty & 0.39 & 0.39 \\ 0.44 & \infty & \infty & \infty \end{bmatrix} \]

This example is feasible from Theorem 2. The minimal solution are \[ X(e_1) = [0.44 \ 0 \ 0.39 \ 0] \] for \( e_1 = [3 \ 1] \) and \[ X(e_2) = [0.44 \ 0 \ 0 \ 0.39] \] for \( e_2 = [4 \ 1] \). In this example, \( e_2 = e^* \) and \( X(e_2) \) is the optimal solution of (*1). Hence,
\[ x^* = [x_1^* \ x_2^* \ x_3^* \ x_4^*] = \begin{bmatrix} \widetilde{X} - \widetilde{X} \ (e_2) \ 2 \ X \ (e_2) \ 3 \ \widetilde{X} \ 4 \end{bmatrix} = [0.58 \ 0 \ 0 \ 0.43] \]
\[ \text{and} \quad c^T x = -0.72. \]

**Remark (2)**

If we convert the objective function of Example2 for instance, into the objective function \( c^T x = -0.5x_1 - 3x_2 + x_3 - x_4 \), then we have
\[ x^* = [x_1^* \ x_2^* \ x_3^* \ x_4^*] = \begin{bmatrix} \widetilde{X} - \widetilde{X} \ (e_2) \ 2 \ X \ (e_2) \ 3 \ \widetilde{X} \ 4 \end{bmatrix} = [0.58 \ 1 \ 0 \ 0.43] \]
\[ \text{and} \quad \inf c^T x = -3.72 \]
instead of \( \min c^T x \).

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