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Optimization of the reducible objective functions with monotone factors subject to FRI constraints defined with continuous t-norms

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Abstract

In this paper, we investigate a special kind of optimization with fuzzy relational inequalities constraints where a continuous t-norm is considered as the fuzzy composition and the objective function can be expressed as $f(x) = h_1(x) \oplus h_2(x)$ in which $h_1(x)$ and $h_2(x)$ are increasing and decreasing functions, respectively, and $\oplus$ is a commutative and monotone binary operator. Some basic properties have been extended a necessary and sufficient condition is presented to realize the feasibility of the problem. Also, an algorithm is given to optimize the objective function on the region of the FRI constraints. Finally, five examples are appended with two continuous t-norms, Lukasiewicz and Yager, and different objective functions, for illustrating.

Keywords: Fuzzy Relational Equations; Fuzzy Relational Inequalities; Fuzzy Compositions; T-norms

Introduction

In this paper, we study the following nonlinear problem:

$$\begin{aligned} \min \quad & f(x) = h_1(x') \oplus h_2(x'') \\ & A o x \le {}^1b \\ & D o x \ge {}^2b \\ & x \in [0,1]^n \end{aligned} \tag{1}$$

where h_1 is an increasing function, h_2 is a decreasing function $\oplus$ is a commutative and monotone binary operator. Also, $I_1 = \{1,2,\dots,m\}$, $I_2 = \{m+1, m+2, \dots, m+l\}$, $J = \{1,2,\dots,n\}$. $A = (a_{ij})_{m\times n}$, $D = (d_{ij})_{l\times n}$ are fuzzy matrices such that $0 \le a_{ij} \le 1$, $\forall i \in I_1$ and $\forall j \in J$, $0 \le d_{ij} \le 1$, $\forall i \in I_2$ and $\forall j \in J$. ${}^1b = ({}^1b_i)_{m\times 1}$ and ${}^2b = ({}^2b_i)_{l\times 1}$ are fuzzy vectors such that $0 \le {}^1b_i \le 1$, $\forall i \in I_1$ and $0 \le {}^2b_i \le 1$, $\forall i \in I_2$, and "o" denotes a continuous t-norm.

If a_i and d_i are the i'th row of matrix A and D, respectively, then problem (1) can be expressed as follows:

$$\begin{aligned} \min \quad & f(x) = h_1(x') \oplus h_2(x'') \\ & a_i o x \le {}^1b_i \qquad i \in I_1 \\ & d_i o x \ge {}^2b_i \qquad i \in I_2 \\ & x \in [0,1]^n \end{aligned} \tag{2}$$

where the constraints mean:

$$a_i o x = \max_{j=1}^{n} \{t(a_{ij}, x_j)\} \le {}^1b_i \tag{3}$$

$$d_i o x = \max_{j=1}^{n} \{t(d_{ij}, x_j)\} \ge {}^2b_i \tag{4}$$

Fuzzy relational equations (FRE) and inequalities (FRI) have been studied by many researchers since the resolution of FRE was proposed by Sanchez [1,2]. Nowadays, it is well-known that many issues associated with a body knowledge can be treated as FRE problems [3]. In addition to such applications, FRE theory has been applied in many fields including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving and FRE [4]. Pedrycz [5] categorized FRE problems in terms of sets under discussion and different operations. Since then, many theoretical improvements have been investigated and many applications have been presented [6-16]. In [17] the author demonstrates how problems of interpolation and approximation of fuzzy functions are converted with solvability of systems of FRE. Also, various types of FRE with continuous triangular norms were used in applied topics [18,19,20]. For example, in [18], the image was divided in blocks and then the FRE with a t-norm were utilized for compressing each block where results obtained using the Lukasiewicz t-norm.

The solvability and the finding of solutions set are the primary (and the most fundamental) subject concerning FRE problems. Many studies have reported fuzzy relational equations with max-min and max-product compositions. Both compositions are special cases of the max-triangular-norm (max-t-norm). If a FRE problem (defined by a continuous t-norm) is nonempty, its feasible solutions set are often a non-convex set including one maximum solution and a finite number of minimal solutions [21]. This non-convexity property is one of two bottlenecks making major contribution to the increase in complexity of problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. On the other hand, Chen and Wang [22,23] proved that a polynomial-time algorithm to find all minimal solutions of FRE (with max-min compositions) may not exist. Also, Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [24]. In fact, the same result holds true for more general t-norms instead of the minimum and product operators [25,26]. Over the last decades, the solvability of FRE defined with different max-t compositions has been investigated by many researches [27-36].

Optimizing an objective function subjected to a system of fuzzy relational equations or inequalities (FRI) is one of the most interesting and on-going topics among the problems related to the FRE (or FRI) theory [37-55,8,9]. By far the most frequently studied aspect is the determination of a minimizer of a linear objective function and the use of the max-min composition [37,56]. So, it is an almost standard approach to translate this type of problem into a corresponding 0-1 integer linear programming problem, which is then solved using a branch and bound method [57,58]. In [59] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider seeking a minimum cost while fulfilling the requirements assumed by a three-tier framework. The topic of the linear optimization problem was also investigated with max-product operation [39,49,60]. Moreover, some studies have determined a more general operator of linear optimization with replacement of max-min and max-product compositions with a max-t-norm composition [44,48,52,54], max-average composition [61,55] or max-star composition [42].

Recently, many interesting generalizations of the linear and non-linear programming problems constrained by FRE or FRI have been introduced and developed based on composite operations and fuzzy relations used in the definition of the constraints, and some developments on the objective function of the problems [62-66,40,53]. For instance, the linear optimization of bipolar FRE was studied by some researchers where FRE was defined with max-min composition [64] and max-Lukasiewicz composition [65,53]. In [65] the authors introduced the optimization problem subjected to a system of bipolar FRE defined as

$$X(A^+,A^-,b)=\{x\in[0,1]^m : x\circ A^+ \vee \tilde{x}\circ A^- = b\}$$

where $\tilde{x}_i = 1-x_i$ for each component of $\tilde{x}=(\tilde{x}_i)_{1\times m}$ and the notations " $\vee$ " and " $\circ$ " denote max operation and the max-Lukasiewicz composition, respectively. They translated the problem into a 0-1 integer linear programming problem which is then solved using well-developed techniques. In [53], the foregoing problem was solved by an analytical method based on the resolution and some structural properties of the feasible region. Ghodousian and khorram [41] studied a mixed fuzzy system formed by two fuzzy relational inequalities $A\varphi x \le b^1$ and $D\varphi x \ge b^2$, where φ is an operator with (closed) convex solutions. Yang [67] studied the optimal solution of minimizing a linear objective function subject to fuzzy relational inequalities where the constraints defined as $a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + ... + a_{in} \wedge x_n \geq b_i$ for $i = 1, \ldots, m$ and $a \wedge b = \min\{a, b\}$. In [40], the authors introduced FRI-FC problem $\min\{ c^T x : A\varphi x^{\circ} b , x \in [0,1]^n \}$, where φ is max-min composition and "$\circ$" denotes the relaxed or fuzzy version of the ordinary inequality "$\leq$".

Another interesting generalization of such optimization problems are related to objective function. A linear fractional programming problem was presented by Wu et al. [66] where FRE defined with max-Archimedean t-norm composition. Dempe and Ruziyeva [62] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [63]. If the objective function is $z(x) = \max_{i=1}^{n} \{\min\{c_i, x_i\}\}$ with $c_i \in [0,1]$, the model is called the latticized problem [68]. Also, Yang et al. [56] introduced another version of the latticized programming problem subject to max-prod fuzzy relation inequalities. On the other hand, Lu and Fang considered the single non-linear objective function and solved it with FRE constraints and max-min operator [69]. They proposed a genetic algorithm for solving the problem. Hassanzadeh et al. [51] used the same GA proposed by Lu and Fang to solve a similar nonlinear problem constrained by FRE and max-product operator. Ghodousian et. al. [46] presented a new genetic algorithm to solve nonlinear problem constrained by Lukasiewicz FRE.

By considering $J = \{1, 2, \ldots, n\}$ suppose that $J', J'' \subseteq J$ such that $J' \cap J'' = \varnothing$ and $J' \cup J'' = J$. Let $|J'| = n_1$, $|J''| = n_2$ and $|J| = n$ denote the cardinality of sets J', J'' and J, respectively. Moreover, consider $J' = \{ j'_1, j'_2, \ldots, j'_{n_1} \}$ and $J'' = \{ j''_1, j''_2, \ldots, j''_{n_2} \}$ in which $j'_k \leq j'_{k+1}$ for each $1 \leq k \leq n_1 - 1$, and $j''_k \leq j''_{k+1}$ for each $1 \leq k \leq n_2 - 1$. Then,

a) $h_1(x_{j'_1}, x_{j'_2}, \ldots, x_{j'_{n_1}}) : R^{n_1} \to R$ and $h_2(x_{j''_1}, x_{j''_2}, \ldots, x_{j''_{n_2}}) : R^{n_2} \to R$ are said to be two functions with distinct variables in J. For example, $h_1(x_1, x_2) = x_1 x_2$ and $h_2(x_3, x_4) = x_3 + x_4$ are two functions with distinct variables in $J = \{1, 2, 3, 4\}$.

b) $h_1(x_{j'_1}, x_{j'_2}, \ldots, x_{j'_{n_1}}) : R^{n_1} \to R$ is said to be an increasing function if $h_1(x_{j'_1}, x_{j'_2}, \ldots, x_{j'_{n_1}})$ is increasing with respect to each of its components, i.e. with respect to each x_j, $\forall j \in J'$.Similarly, $h_2(x_{j''_1}, x_{j''_2}, \ldots, x_{j''_{n_2}}) : R^{n_2} \to R$ is said to be a decreasing function if

c) $h_2(x_{j''_1}, x_{j''_2}, \ldots, x_{j''_{n_2}})$ is decreasing with respect to each of its components, i.e. with respect to each x_j, $\forall j \in J''$. Also, a function f is said to be monotone if f is either an increasing function or a decreasing function.

A function $f:[0,1]^n \to [0,1]$ is said to be a reducible function with monotone factors if and only if at least one of the following statements holds ;

I. there exists an increasing function $h_1:[0,1]^{n_1} \to [0,1]$ such that $n_1 = n$ and $h_1 = f$. In this case, f has a single factor h_1.

II. there exists a decreasing function $h_2:[0,1]^{n_2} \to [0,1]$ such that $n_2 = n$ and $h_2 = f$. In this case, f has a single factor h_2.

III. there exist an increasing function $h_1:[0,1]^{n_1} \to [0,1]$ and a decreasing function $h_2:[0,1]^{n_2} \to [0,1]$ with distinct variables in J such that $f = h_1 \oplus h_2$ in which $\oplus:[0,1]^2 \to [0,1]$ is a commutative and monotone binary operator, i.e.

$$\forall a,b \in [0,1]:\ a\oplus b = b\oplus a$$
$$\forall a,b,c,d \in [0,1]: \text{if } a \le c \text{ and } b \le d \Rightarrow a\oplus b \le c\oplus d$$

In case III, f has two factors h_1 and h_2, and we can convert each vector $x = (x_1, x_2, ..., x_n)$ into two sub-vectors $x' = (x_{j'_1}, x_{j'_2}, ..., x_{j'_{n_1}})$ and $x'' = (x_{j''_1}, x_{j''_2}, ..., x_{j''_{n_2}})$ such that $f(x) = h_1(x') \oplus h_2(x'')$.

In this paper, we study optimizing the reducible objective function with monotone factors subject to the general FRI constraints in which an arbitrary continuous t-norm is used to define the fuzzy relational inequalities. Also, we compare some relationships between the general results obtained in this paper and similar results in the special cases obtained (special cases either with regard to their objective functions such as linear programming, or regarding their feasible regions such as FRE or FRI constraints defined with particular continuous t-norms) in some previous papers.

The remainder of the paper is organized as follows. Section 2 take a brief look at some basic results on the feasible region of problem (2) in the general case. In section 3, a modification operation is given to accelerate the resolution process of the problem. Also, the necessary and sufficient condition is presented to realize whether problem (2) is feasible or not. In section 4, problem (2) is completely resolved. Finally, we give five examples in section 5.

Some basic properties of problem

Definition (1): For each $i \in I_1$, we set

$${}^iS_1(a_i, {}^1b_i) = \{x \in [0,1]^n : a_i o x \le {}^1b_i\}.$$

Similarly, for each $i \in I_2$, we define ${}^iS_2(d_i, {}^2b_i) = \{x \in [0,1]^n : d_i o x \le {}^2b_i\}$. Also,

$$S_1(A, {}^1b) = \{x \in [0,1]^n : A o x \le {}^1b\},$$

$$S_2(D, {}^2b) = \{x \in [0,1]^n : D o x \le {}^2b\}.$$

Furthermore,

$$S(A, D, {}^1b, {}^2b) = \{x \in [0,1]^n : A o x \le {}^1b, D o x \ge {}^2b\}$$

Definition (2): We set

$${}^t x_{\max}(a,b) = \sup\{x \in [0,1]^n : t(a,x) \le b\}$$

.By the identity law and the monotonicity properties of t-norms, ${}^t x_{\max}(a,b) = 1$ where

$a \le b$. Moreover, from relation (3) and the identity law of t-norms, the vector $\bar{0} = [0,0,\ldots,0]_{1\times n}$ is obviously the single minimum solution of the inequality $a_i \, o \, x \le {}^1b_i$, $\forall i \in I_1$. Thus, the zero vector $\bar{0}$ is the single minimum solution for $A \, o \, x \le {}^1b$. Also, Definition 3 below gives the single maximum solution of the inequality $a_i \, o \, x \le {}^1b_i$, for each fixed $i \in I_1$.

Definition (3): Define ${}^i\hat{x} = [\,{}^i\hat{x}_1, {}^i\hat{x}_2, \ldots, {}^i\hat{x}_n\,]$ where ${}^i\hat{x}_j = {}^tx_{\max}(a_{ij}, {}^1b_i)$.

Corollary (1): ${}^iS_1(a_i, {}^1b_i) = [\bar{0}, {}^i\hat{x}]$, $\forall i \in I_1$.

Proof. See Remark 2.5 in [15]. ❐

Lemma (1): $S_1(A, {}^1b) = [\bar{0}, \hat{x}]$ where $\hat{x} = \min_{i=1}^{m} \{ {}^i\hat{x} \}$.

Proof: The proof is easily obtained from Corollary 1 and the equation $S_1(A, {}^1b) = \bigcap_{i \in I_1} {}^iS(a_i, {}^1b_i)$. ❐

It can be shown by (4), the identity law and the monotonicity of t-norms that for each $i \in I_2$, the inequality $d_i \, o \, x \ge {}^2b_i$ has a feasible solution iff there exists at least some $j \in J$ such that $d_{ij} \ge {}^2b_i$. Also, if this condition is satisfied then $\bar{1} = [1,1,\ldots,1]_{1\times n}$ is the single maximum solution of the inequality $d_i \, o \, x \ge {}^2b_i$. Then, we have Corollary 2 below.

Corollary (2): For each $i \in I_2$, ${}^iS_2(d_i, {}^2b_i) \ne \varnothing$ if there exists at least some $j \in J$ such that $d_{ij} \ge {}^2b_i$. Also, $\bar{1} \in {}^iS_2(d_i, {}^2b_i)$ is the single maximum solution iff ${}^iS_2(d_i, {}^2b_i) \ne \varnothing$.

Definition (4): We set

$$ {}^tx_{\min}(a,b) = \begin{cases} \inf\{x \in [0,1]^n : t(a,x) \ge b\} & a \ge b \\ \infty & a < b \end{cases} $$

From Definition 4, we have ${}^tx_{\min}(a,b) \ge b$ if $a \ge b$. Because, if $a \ge b$ and ${}^tx_{\min}(a,b) < b$, we have

$$ t(a, {}^tx_{\min}(a,b)) \le t(1, {}^tx_{\min}(a,b)) = {}^tx_{\min}(a,b) < b $$

which contradicts $t(a, {}^tx_{\min}(a,b)) \ge b$.

Definition (5): Let $J_i = \{ j \in J : d_{ij} \ge {}^2b_i \}$. For each $i \in I_2$ and $j \in J_i$ we define ${}^i\check{x}(j) = [\,{}^i\check{x}(j)_1, {}^i\check{x}(j)_2, \ldots, {}^i\check{x}(j)_n\,]$ where

$$ {}^i\check{x}(j)_k = \begin{cases} {}^tx_{\min}(d_{ij}, {}^2b_i) & k = j \\ 0 & k \ne j \end{cases} $$

By (4) and Definitions 5 and 6, ${}^i\check{x}(j)$ is the feasible minimal point in the set ${}^iS_2(d_i, {}^2b_i)$, $\forall i \in I_2$ and $j \in J_i$. Also, we have Corollary 3 below that is proved in a similar manner to that of Corollary 1.

Corollary (3): If ${}^{i}S_2(d_i, {}^{2}b_i) \neq \varnothing$, then ${}^{i}S_2(d_i, {}^{2}b_i) = \bigcup_{j \in J_i} [\, {}^{i}\check{x}(j), \bar{1}\,]$, $\forall i \in I_2$

Definition (6): Let $e = (e(1), e(2), \ldots, e(l))$ such that $e(i) = j \in J_i$, $\forall i \in I_2$, and let E be the set of all the vectors e. We define $\check{x}(e) = \max_{i \in I_2} \{ {}^{i}\check{x}(e(i)) \}$

Lemma (2): $S_2(D, {}^{2}b) = \bigcup_{e \in E} [\, \check{x}(e), \bar{1}\,]$.

Proof. The proof is obtained in a similar way to that of Lemma 1 using Corollary 3 and Definition 6. ❒

Remark (1): From Corollary 2 or Lemma 2, $S_2(D, {}^{2}b) \neq \varnothing$ if $\bar{1} \in S_2(D, {}^{2}b)$. Also, if there exist some $i \in I_2$ and $j \in J_i$ such that ${}^{t}x_{\min}(d_{ij}, {}^{2}b_j) = 0$, we have ${}^{i}S_2(d_i, {}^{2}b_i) = [\bar{0}, \bar{1}]^n$. Therefore, in this case, we can remove the ith row of matrix D without any effect on the set of solutions.

From Lemmas 1 and 2, $S(A, D, {}^{1}b, {}^{2}b) \neq \varnothing$ if

$$S_1(A, {}^{1}b) \bigcap S_2(D, {}^{2}b) = [\bar{0}, \hat{x}] \bigcap \Big(\bigcup_{e \in E} [\, \check{x}(e), \bar{1}\,] \Big) \neq \varnothing$$

which means $S(A, D, {}^{1}b, {}^{2}b) \neq \varnothing$ iff $S_2(D, {}^{2}b) \neq \varnothing$ and there exists at least some $e \in E$ such that $\check{x}(e) \leq \hat{x}$. The following theorem formulates this fact and determines the shape of the set of feasible solutions.

Theorem (1):

(a) $S(A, D, {}^{1}b, {}^{2}b) \neq \varnothing$ iff $\hat{x} \in S_2(D, {}^{2}b)$.

(b) If $S(A, D, {}^{1}b, {}^{2}b) \neq \varnothing$ then $S(A, D, {}^{1}b, {}^{2}b) = \bigcup_{e \in E'} [\, \check{x}(e), \hat{x}\,]$ where $E' = \{ e \in E : \check{x}(e) \leq \hat{x} \}$.

Proof: See Theorems 2.3 and 3.1 in [15]. ❒

From Theorem 1, if $S(A, D, {}^{1}b, {}^{2}b) \neq \varnothing$ then it is determined by the single maximum solution and finite minimal solutions. Also, in order to find the minimal solutions it is sufficient to consider only vectors $e \in E'$. Actually, some selections of the vectors e result in generating vectors $\check{x}(e)$ which are infeasible. For an efficient method, it would be better to realize the vectors e constructing the feasible vectors $\check{x}(e)$ before generating the solutions $\check{x}(e)$. Such vectors are distinguished by the property stated in Lemma 3 below.

Lemma (3): $e \in E'$ iff ${}^{t}x_{\min}(d_{i\,e(i)}, {}^{2}b_i) \leq \hat{x}_{e(i)}$, $\forall i \in I_2$.

Proof: Suppose $e \in E'$ and $e(i') = j'$ for a fixed arbitrary $i' \in I_2$. Since $e \in E'$ we have $\check{x}(e) = \max_{i \in I_2} \{ {}^{i}\check{x}(e(i)) \} \leq \hat{x}$, which implies :

$$\check{x}(e)_{j'} = \max_{i \in I_2} \{ {}^{i}\check{x}(e(i))_{j'} \} \leq \hat{x}_{j'} \qquad \textbf{(5)}$$

Let $I_{j'}(e)=\{i \in I_2 : e(i)=j'\}$. Then, from Definitions 5 and 6 and the fact that $e(i)=j'$ for each $i \in I_{j'}(e)$, we have ${}^{i}\check{x}(e(i))_{j'}=0$, $\forall i \notin I_{j'}(e)$, and

${}^{i}\check{x}(e(i))_{j'} = {}^{i}\check{x}(j')_{j'} = {}^{t}x_{\min}(d_{ij'}, {}^{2}b_i) = {}^{t}x_{\min}(d_{ie(i)}, {}^{2}b_i)$, $\forall i \in I_{j'}(e)$

Therefore, (5) is converted into :

$$\check{x}(e)_{j'} = \max_{i \in I_{j'}(e)} \{ {}^{t}x_{\min}(d_{ie(i)}, {}^{2}b_i)\} \le \hat{x}_{j'} \quad \textbf{(6)}$$

Relations (6) require ${}^{t}x_{\min}(d_{i\,e(i)}, {}^{2}b_i) \le \hat{x}_{e(i)}$, $\forall i \in I_{j'}(e)$. Now, since $i' \in I_{j'}(e)$ then ${}^{t}x_{\min}(d_{i'\,e(i')}, {}^{2}b_{i'}) \le \hat{x}_{e(i')}$. Since this inequality holds for each $i' \in I_2$, then the necessary condition is proved. Conversely, suppose ${}^{t}x_{\min}(d_{i\,e(i)}, {}^{2}b_i) \le \hat{x}_{e(i)}$, $\forall i \in I_2$, and fix an arbitrary $j \in J$. If $I_j(e)=\varnothing$ then $e(i) \neq j$, $\forall i \in I_2$, which implies ${}^{i}\check{x}(e(i))_j = 0$, $\forall i \in I_2$.

Then,

$$\check{x}(e)_j = 0 \le \hat{x}_j \quad \textbf{(7)}$$

Otherwise, if $I_j(e)\neq\varnothing$, we have

$$\check{x}(e)_j = \max_{i\in I_j(e)}\{ {}^{i}\check{x}(e(i))_j\} = \max_{i\in I_j(e)}\{ {}^{i}\check{x}(j)_j\} = \max_{i\in I_j(e)}\{ {}^{t}x_{\min}(d_{ij}, {}^{2}b_i)\}$$
$$= \max_{i\in I_j(e)}\{ {}^{t}x_{\min}(d_{ie(i)}, {}^{2}b_i)\}$$

Now, since we have ${}^{t}x_{\min}(d_{i\,e(i)}, {}^{2}b_i) \le \hat{x}_{e(i)}$, $\forall i \in I_2$, from the assumption, then ${}^{t}x_{\min}(d_{i\,e(i)}, {}^{2}b_i) \le \hat{x}_{e(i)} = \hat{x}_j$, $\forall i \in I_j(e) \subseteq I_2$, which requires $\max_{i\in I_j(e)}\{ {}^{t}x_{\min}(d_{ie(i)}, {}^{2}b_i)\} \le \hat{x}_j$. Therefore,

$$\check{x}(e)_j \le \hat{x}_j \quad \textbf{(8)}$$

Since Inequalities (7) and (8) are satisfied for each $j \in J$, then $\check{x}(e) \le \hat{x}$, which implies $e \in E'$. Then, the sufficient condition is proved. ❒

Corollary (4): In order to accelerate the finding process of the solutions $\check{x}(e)$ for each $i \in I_2$, we can reduce set J_i by removing each $j \in J_i$ such that ${}^{t}x_{\min}(d_{ij}, {}^{2}b_i) > \hat{x}_j$, before selecting the vectors e to construct such solutions as $\check{x}(e)$.

By Lemma 3 and Corollary 2, the conditions for selecting $e \in E'$ are $d_{ij} \ge {}^{2}b_i$ and ${}^{t}x_{\min}(d_{ij}, {}^{2}b_i) \le \hat{x}_j$, for each $i \in I_2$ and $j \in J$. If the max-min composition is used as a continuous t-norm in Problem (2), then the above conditions will be converted into $d_{ij} \ge {}^{2}b_i$ and $\hat{x}_j \ge {}^{2}b_i$ or, equivalently, into $\hat{x}_j \wedge d_{ij} \ge {}^{2}b_i$ used in the definition of the characteristic matrix $C=(c_{ij})_{m\times n}$ and Theorem 2.2 in [8].

Definition (7): Let $M=(m_{ij})_{l\times n}$ be a matrix whose components are defined as follows for each $i\in I_2$ and each $j\in J$

$$m_{ij}=\begin{cases} {}^{t}x_{\min}(d_{ij}, {}^{2}b_i) & d_{ij}\geq {}^{2}b_i \\ \infty & otherwise \end{cases}$$

We construct the modified matrix $M^*=(m^*_{ij})_{l\times n}$ from matrix M as follows:

$$m^*_{ij}=\begin{cases} \infty & d_{ij}\geq {}^{2}b_i \ and \ {}^{t}x_{\min}(d_{ij}, {}^{2}b_i)>\hat{x}_j \\ m_{ij} & otherwise \end{cases}$$

Corollary (5): By the definition of matrix M^*, if we set $J_i^*=\{j\in J: m_{ij}^*\neq\infty\}$ and $E^*=\{e\in E: e(i)\in J_i^*, \forall i\in I_2\}$ then we have $J_i^*\subseteq J_i$ and $E'=E^*\subseteq E$.

If equalities replace inequalities in problem (2) then matrix M^* above will be converted into one that is in agreement with the definition of the minimal solution matrix $\check{\Gamma}$ in [70]. Also, if equalities and max-min composition replace inequalities and general continuous t-norms, respectively, then Definition 7 will be equivalent to the well-known simplified matrix obtained from the equivalence operation defined by Lu and Fang in Definition 1 and Lemma 2 in [69]. To prove that the simplified matrix defined by Lu and Fang is actually matrix M^* and, equivalently, to prove that the foregoing simplified matrix is equivalent to removing vectors e resulting in infeasible vectors $\check{x}(e)$, we refer the readers to Lemmas 1 and 3 in [39] in which the similar simplified matrix has been used with max-product composition.

Theorem (2):

(a) The i th row of matrix D can be removed without any effect on the set of solutions if $m_{ij}=0$ for at least some $j\in J$.

(b) $S(A,D,{}^{1}b,{}^{2}b)=\varnothing$ if there exists at least one $i\in I_2$ such that $m_{ij}=\infty$, $\forall j\in J$.

(c) $S(A,D,{}^{1}b,{}^{2}b)\neq\varnothing$ iff for each $i\in I_2$ there exists at least one $j_i\in J$ such that $m^*_{ij_i}\neq\infty$.

Proof: (a) and (b) are obtained from Remark 1 and Corollary 2, respectively. (c) From part (b) of Theorem 1, if $S(A,D,{}^{1}b,{}^{2}b)\neq\varnothing$ then $E'\neq\varnothing$. Fix a vector $e'\in E'$. By considering Lemma 3, we have ${}^{t}x_{\min}(d_{i\,e'(i)}, {}^{2}b_i)\leq\hat{x}_{e'(i)}$, $\forall i\in I_2$. Thus, from Definitions 7 and 8 we have $m^*_{i\,e'(i)}\neq\infty$, $\forall i\in I_2$. Then, for each $i\in I_2$ there exists $j_i=e'(i)$ such that $m^*_{ij_i}\neq\infty$. Conversely, suppose for each $i\in I_2$ there exists at least one $j_i\in J$ such that $m^*_{ij_i}\neq\infty$. From the definition of matrix M^*, we certainly have $j_i\in J_i$ and ${}^{t}x_{\min}(d_{ij_i}, {}^{2}b_i)\leq\hat{x}_{j_i}$

For the vector $e''=(e''(1),e''(2),\dots,e''(m))$, we set $e''(i)=j_i$. From (5) and Definition 6, $\check{x}(e'')\leq\hat{x}$. Thus $e''\in E'$ and

$\bigcup_{E'} [\,\check{x}(e), \hat{x}\,] \neq \varnothing$, which implies $S(A, D, {}^1b, {}^2b) \neq \varnothing$ by Theorem 1. ❐

Theorem 3 below shows that for finding the vectors e for generating the feasible solutions $\check{x}(e)$, we can use E^* $(= E')$ instead of E as the search domain, which accelerates the finding process of the feasible solutions $\check{x}(e)$.

Theorem (3): Suppose $S(A, D, {}^1b, {}^2b) \neq \varnothing$. $e \in E^*$ iff $\check{x}(e) \in S(A, D, {}^1b, {}^2b)$.

Proof: Suppose $e \in E^*$. Then we have $e(i) \in J_i^*$, $\forall i \in I_2$. Also, $m^*_{ie(i)} \neq \infty$, $\forall i \in I_2$, which implies ${}^t x_{\min}(d_{i\,e(i)}, {}^2b_i) \leq \hat{x}_{e(i)}$, $\forall i \in I_2$. Now, by considering a fixed $j \in J$, we have $\check{x}(e)_j = \max_{i \in I_2}\{\,{}^i\check{x}(e(i))_j\,\} = \max_{i \in I_j(e)}\{\,{}^i\check{x}(e(i))_j\,\} = \max_{i \in I_j(e)}\{\,{}^t x_{\min}(d_{ie(i)}, {}^2b_i)\}$ $\leq \hat{x}_{e(i)} = \hat{x}_j$.

Since these inequalities are satisfied for each $j \in J$, then $\check{x}(e)_j \leq \hat{x}_j$, $\forall j \in J$, which implies $\check{x}(e) \leq \hat{x}$ and then $\check{x}(e)$ $\in S(A, D, {}^1b, {}^2b)$ from part (b) of Theorem 1. Conversely, suppose $e \notin E^*$. Then there exists some $i' \in I_2$ such that $e(i') \notin J_i^*$. Therefore $m^*_{i'e(i')} = \infty$ which implies ${}^t x_{\min}(d_{i'e(i')}, {}^2b_{i'}) > \hat{x}_{e(i')}$. Now, we have:

$\check{x}(e)_{e(i')} = \max_{i \in I_2}\{\,{}^i\check{x}(e(i))_{e(i')}\,\} \geq {}^{i'}\check{x}(e(i'))_{e(i')} = {}^t x_{\min}(d_{i'e(i')}, {}^2b_{i'}) > \hat{x}_{e(i')}$. Hence, $\check{x}(e) \nleq \hat{x}$, which requires $\check{x}(e) \notin S(A, D, {}^1b, {}^2b)$ by Theorem 1, part(b). ❐

Optimizing reducible functions with monotone factors

As mentioned, a reducible function with monotone factors can be interpreted as cases I, II and III stated in the first section. Clearly, I and II are special cases of III. In case III, we convert problem (2) into two sub-problems as follows:

$$
\begin{aligned}
\min\ & h_1(x) \\
& A \circ x \leq {}^1b \\
& D \circ x \geq {}^2b \\
& x \in [0,1]^n
\end{aligned}
\qquad \textbf{(*1)}
$$

$$
\begin{aligned}
\min\ & h_2(x) \\
& A \circ x \leq {}^1b \\
& D \circ x \geq {}^2b \\
& x \in [0,1]^n
\end{aligned}
\qquad \textbf{(*2)}
$$

The following theorem gives the optimal solution of problem (2) in each case.

Theorem (4): Consider problem (2) and suppose that f is defined on $S(A, D, {}^1b, {}^2b)$.

a) If there exists an increasing function $h_1:[0,1]^{n_1} \to [0,1]$ such that $n_1 = n$ and $h_1 = f$, then $\check{x}(e^*)$ is the optimal solution of problem (2) for some $e^* \in E'$.

b) If there exists a decreasing function $h_2:[0,1]^{n_2} \to [0,1]$ such that $n_2 = n$ and $h_2 = f$, then $\hat{x}$ is the optimal solution of problem (2).

c) Suppose that there exist an increasing function $h_1:[0,1]^{n_1} \to [0,1]$ and a decreasing function $h_2:[0,1]^{n_2} \to [0,1]$ with distinct variables such that $f(x_1,x_2,...,x_n) = h_1(x_{j'_1},x_{j'_2},...,x_{j'_{n_1}}) \oplus h_2(x_{j''_1},x_{j''_2},...,x_{j''_{n_2}})$ where $\oplus$ is a commutative and monotone binary operator. If $\check{x}(e^*)$ is the optimal solution of (*1), then $x^* = (x_1^*, x_2^*, ..., x_n^*)$ is the optimal solution of problem (2), where

$$x_j^* = \begin{cases} \hat{x}_j & j \in J'' \\ \check{x}(e^*)_j & j \in J' \end{cases} \qquad j = 1,2,...,n$$

$J' = \{ j'_1, j'_2, ..., j'_{n_1} \}$ and $J'' = \{ j''_1, j''_2, ..., j''_{n_2} \}$.

Proof:

a) Set $h_1(\check{x}(e^*)) = \min\{h_1(\check{x}(e)) : e \in E'\}$.

We show $\check{x}(e^*)$ is the optimal solution of (*1). Consider an arbitrary point $\tilde{x} \in S(A,D,{}^1b,{}^2b)$. If $\tilde{x} \in \{x \in [0,1]^n : x = \check{x}(e)\ \textit{for some}\ e \in E'\}$, then the proof is readily attained from $h_1(\check{x}(e^*)) = \min\{h_1(\check{x}(e)) : e \in E'\}$. Otherwise, suppose there is no $e \in E'$ such that $\tilde{x} = \check{x}(e)$. In this case, since $\tilde{x} \in S(A,D,{}^1b,{}^2b)$ then by Theorem 1 we have $\check{x}(\tilde{e}) \le \tilde{x}$ for some $\tilde{e} \in E'$. Now, since h_1 is increasing, this inequality together with $h_1(\check{x}(e^*)) = \min\{h_1(\check{x}(e)) : e \in E'\}$ imply $f(\check{x}(e^*)) = h_1(\check{x}(e^*)) \le h_1(\check{x}(\tilde{e})) \le h_1(\tilde{x})$

Thus, $f(\check{x}(e^*)) \le f(x)$, $\forall x \in S(A,D,{}^1b,{}^2b)$, which proves the statement.

b) By Theorem1 $x \le \hat{x}$, $\forall x \in S(A,D,{}^1b,{}^2b)$. Now, since $h_2 = f$ is a decreasing function, we have $f(\hat{x}) \le f(x)$, $\forall x \in S(A,D,{}^1b,{}^2b)$, which proves the statement.

c) Suppose $\tilde{x} \in S(A,D,{}^1b,{}^2b)$. By the assumptions there exist $J' = \{ j'_1, j'_2, ..., j'_{n_1} \}$ and $J'' = \{ j''_1, j''_2, ..., j''_{n_2} \}$ such that $J', J'' \subseteq J$, $J' \cap J'' = \varnothing$ and $J' \cup J'' = J$. Also, $h_1(x_{j'_1}, x_{j'_2}, ..., x_{j'_{n_1}}):[0,1]^{n_1} \to [0,1]$ is an increasing function and $h_2(x_{j''_1}, x_{j''_2}, ..., x_{j''_{n_2}}):[0,1]^{n_2} \to [0,1]$ is a decreasing function such that $f(x) = h_1(x') \oplus h_2(x'')$. By Theorem 1 we have $\check{x}(\tilde{e}) \le \tilde{x} \le \hat{x}$ for some $\tilde{e} \in E'$. Then,

$\check{x}(\tilde{e})_j \le \tilde{x}_j \le \hat{x}_j$, $\forall j \in J$. Especially, $\check{x}(\tilde{e})_j \le \tilde{x}_j$, $\forall j \in J'$, and $\tilde{x}_j \le \hat{x}_j$, $\forall j \in J''$. Then, since $h_1(x_{j'_1}, x_{j'_2}, \dots, x_{j'_{n_1}})$ is increasing, we have :

$$h_1(\check{x}(e^*)) \le h_1(\check{x}(\tilde{e})_{j'_1}, \dots, \check{x}(\tilde{e})_{j'_{n_1}}) \le h_1(\tilde{x}_{j'_1}, \dots, \tilde{x}_{j'_{n_1}}) \quad \textbf{(*)}$$

The first inequality is attained from the optimality of $\check{x}(e^*)$ in (*1). Similarly, since $h_2(x_{j''_1}, x_{j''_2}, \dots, x_{j''_{n_2}})$ is decreasing, we have :

$$h_2(\hat{x}_{j''_1}, \dots, \hat{x}_{j''_{n_2}}) \le h_2(\tilde{x}_{j''_1}, \dots, \tilde{x}_{j''_{n_2}}) \quad \textbf{(**)}$$

Thus, from (*) , (**) and the monotonocity of operator $\oplus$, we have

$$h_1(\check{x}(e^*)) \oplus h_2(\hat{x}_{j''_1}, \dots, \hat{x}_{j''_{n_2}}) \le$$
$$h_1(\tilde{x}_{j'_1}, \dots, \tilde{x}_{j'_{n_1}}) \oplus h_2(\tilde{x}_{j''_1}, \dots, \tilde{x}_{j''_{n_2}}) \quad \textbf{(***)}$$

But, from Remark 1

$f(x^*) = h_1(\check{x}(e^*)) \oplus h_2(\hat{x}_{j''_1}, \dots, \hat{x}_{j''_{n_2}})$ and $f(\tilde{x}) = h_1(\tilde{x}_{j'_1}, \dots, \tilde{x}_{j'_{n_1}}) + h_2(\tilde{x}_{j''_1}, \dots, \tilde{x}_{j''_{n_2}})$.

These equalities together with (***) imply $f(x^*) \le f(\tilde{x})$, which proves the statement. ❒

As a special case of the discussions presented so far, when $f(x)$, $h_1(x)$ and $h_2(x)$ are linear functions, see [41,71,72] in which we can write by our notation : $f(x) = \sum_{j=1}^{n} c_j x_j$, $h_1(x) = \sum_{j=1}^{n} c_j^+ x_j$ and $h_2(x) = \sum_{j=1}^{n} c_j^- x_j$, where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \min\{c_j, 0\}$ for $j = 1, 2, \dots, n$. Also, as special case of Theorem 4, see Theorem 2.1 in [22].

The algorithm below abbreviates the solution steps of Problem (2).

Algorithm. Given problem (2),

1. Calculate ${}^i\hat{x}$, $\forall i \in I_1$ by Definition 3.

2. Calculate $\hat{x} = \min_{i=1}^{m} \{ {}^i\hat{x} \}$.

3. Calculate matrix M by Definition 7. If there exists at least some $i \in I_2$ such that $m_{ij} = \infty$, $\forall j \in J$, then stop; the problem is infeasible. If there exists $i \in I_2$ such that $m_{ij} = 0$ for some $j \in J_i$ remove row i.

4. Calculate matrix M^* by Definition 7. If there exists some $i \in I_2$ such that $m^*_{ij} = \infty$, $\forall j \in J$, then stop. The problem is infeasible.

5. Select the vectors $e \in E''$ and calculate the vectors $\check{x}(e)$ from matrix M^*.

6. Find the optimal solution of (*1) from the vector $\check{x}(e)$ obtained in step 5.

7. Calculate the optimal solution of Problem (2) by Theorem 4.

Numerical examples:

Example (1): Consider the problem below with the Lukasiewicz t-norm

$$\min \ f(x)=3x_1-4x_2+x_3-2x_4$$

$$\begin{bmatrix} 0.96 & 0.9 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.25 & 0.75 \\ 0.2 & 0.5 & 0.34 & 0.8 \end{bmatrix} o\, x \le \begin{bmatrix} 0.9 & 0.5 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 0.93 & 0.41 & 0.8 & 0.2 \\ 0.46 & 0.78 & 0.63 & 0.55 \\ 0.32 & 0.87 & 0.32 & 0.1 \end{bmatrix} o\, x \ge \begin{bmatrix} 0.82 & 0.43 & 0.25 \end{bmatrix}$$

$$x \in [0,1]^n$$

where $x\, o\, y = \max\{x+y-1,0\}$. In this example, we have

${}^{1}\hat{x}=\begin{bmatrix} 0.94 & 1 & 1 & 1 \end{bmatrix}$, ${}^{2}\hat{x}=\begin{bmatrix} 1 & 1 & 1 & 0.75 \end{bmatrix}$, ${}^{3}\hat{x}=\begin{bmatrix} 1 & 0.8 & 0.96 & 0.5 \end{bmatrix}$ and then the maximum solution is $\hat{x}=\begin{bmatrix} 0.94 & 0.8 & 0.96 & 0.5 \end{bmatrix}$. Also, we have

$$M=\begin{bmatrix} 0.89 & \infty & \infty & \infty \\ 0.97 & 0.65 & 0.8 & 0.88 \\ 0.93 & 0.38 & 0.93 & \infty \end{bmatrix}$$

$$M^*=\begin{bmatrix} 0.89 & \infty & \infty & \infty \\ \infty & 0.65 & 0.8 & \infty \\ 0.93 & 0.38 & 0.93 & \infty \end{bmatrix}$$

This example is feasible by considering matrix M^* by part (c) of Theorem 2. By selecting the vectors e from matrix M^*, the vectors $\check{x}(e)$ are obtained as follows such that $e \in E''$.

${}^{1}\check{x}=\begin{bmatrix} 0.89 & 0.65 & 0 & 0 \end{bmatrix}$,

${}^{2}\check{x}=\begin{bmatrix} 0.93 & 0 & 0.8 & 0 \end{bmatrix}$,

${}^{3}\check{x}=\begin{bmatrix} 0.89 & 0.38 & 0.8 & 0 \end{bmatrix}$,

${}^{4}\check{x}=\begin{bmatrix} 0.89 & 0 & 0.93 & 0 \end{bmatrix}$.

The two feasible vectors ${}^{5}\check{x}=\begin{bmatrix} 0.93 & 0.65 & 0 & 0 \end{bmatrix}$ and ${}^{6}\check{x}=\begin{bmatrix} 0.89 & 0.65 & 0.93 & 0 \end{bmatrix}$ are not actually the minimal solutions. However, they are removed by pairwise comparison.

Furthermore, $J'=\{1,3\}$, $J''=\{2,4\}$, $x'=(x_1,x_3)$ and $x''=(x_2,x_4)$. Also, $h_1:[0,1]^2 \to [0,1]$ and $h_2:[0,1]^2 \to [0,1]$ where $h_1(x_1,x_3)=3x_1+x_3$ is an increasing function and $h_2(x_2,x_4)=-4x_2-2x_4$ is a decreasing function. Finally, $f(x)=h_1(x')+h_2(x'')$ is a reducible function with monotone factors. By comparing the values of the objective function at the minimal solutions in sub-problem (*1), we find $\check{x}(e^*)={}^{1}\check{x}$. Thus, from Theorem 4 :

$$x^*=[x_1^*,x_2^*,x_3^*,x_4^*]=[{}^{1}\check{x}_1,\hat{x}_2,{}^{1}\check{x}_3,\hat{x}_4]=\begin{bmatrix} 0.89 & 0.8 & 0 & 0.5 \end{bmatrix}$$

and then $f(x^*)=-1.53$

For more examples on the linear programming problems subject to FRE or FRI constraints, see also [41,43,45,71,72].

Example (2): With the same assumptions as in Example 1, we define

$$f(x)=(3\wedge x_1^{-\frac{1}{2}})\vee(1\wedge x_2^{4})\vee(12\wedge x_3^{-2})\vee(6\wedge x_4^{-7})$$

where "$\wedge$" and "$\vee$" denote min operator and max operator, respectively (see also [38]). For this example, $J'=\{2\}$, $J''=\{1,3,4\}$, $x'=x_2$ and $x''=(x_1,x_3,x_4)$. Then, we have $h_1:[0,1]\to[0,1]$ and $h_2:[0,1]^3\to[0,1]$ where $h_1(x_2)=(1\wedge x_2)$ an increasing is function and $h_2(x_1,x_3,x_4)=(3\wedge x_1^{-\frac{1}{2}})\vee(12\wedge x_3^{-2})\vee(6\wedge x_4^{-7})$ is a decreasing function. Finally, $f(x)=h_1(x')\vee h_2(x'')$ is a reducible function with monotone factors. By comparing the values of the objective function at the minimal solutions in sub-problem (*1), we have $\check{x}(e^*)={}^{2}\check{x}(\ or\ {}^{4}\check{x})$. Thus, from Theorem 4 we have

$$x^*=[x_1^*,x_2^*,x_3^*,x_4^*]=[\hat{x}_1,{}^{2}\check{x}_2(or\ {}^{4}\check{x}_2),\hat{x}_3,\hat{x}_4]=\begin{bmatrix}0.94 & 0 & 0.96 & 0.5\end{bmatrix}$$

and then $f(x^*)=6$.

Example (3): With the same assumptions as in Example 1, we define $f(x)=\dfrac{2x_1+3x_3}{x_4+x_2+1}$

So, $J'=\{1,3\}$, $J''=\{2,4\}$, $x'=(x_1,x_3)$ and $x''=(x_2,x_4)$. Also, $h_1:[0,1]^2\to[0,1]$ and $h_2:[0,1]^2\to[0,1]$ where $h_1(x_1,x_3)=2x_1+3x_3$ is an increasing function and $h_2(x_2,x_4)=\dfrac{1}{x_4+x_2+1}$ is a decreasing function. Finally, $f(x)=h_1(x')h_2(x'')$ is a reducible function with monotone factors. By comparing the values of the objective function at the minimal solutions in sub-problem (*1), we find $\check{x}(e^*)={}^{1}\check{x}$. Thus, from Theorem 4 we obtain

$$x^*=[x_1^*,x_2^*,x_3^*,x_4^*]=[{}^{1}\check{x}_1,\hat{x}_2,{}^{1}\check{x}_3,\hat{x}_4]=\begin{bmatrix}0.89 & 0.8 & 0 & 0.5\end{bmatrix}$$

and then $f(x^*)=3.173333333$

Example (4): Consider the problem below with the Yager t-norm :

$$\min\ cx=0.5x_1+3x_2+x_3-x_4$$

$$\begin{bmatrix}0.89 & 0.35 & 0.4 & 1\\ 0.62 & 1 & 1 & 0.42\end{bmatrix} o\,x\le\begin{bmatrix}0.4 & 0.63\end{bmatrix}$$

$$\begin{bmatrix}0.5 & 0.4 & 1 & 0.13\\ 0.2 & 1 & 0.2 & 0.2\\ 0.11 & 0.03 & 1 & 0.3\\ 0.1 & 1 & 1 & 0.8\end{bmatrix} o\,x\ge\begin{bmatrix}0.4 & 0.2 & 0.32 & 0.6\end{bmatrix}$$

$$x\in[0,1]^n$$

where

$$x\,o\,y=1-\min\{1,((1-x)^p+(1-y)^p)^{\frac{1}{p}}\},\ p\ge 1$$

. In this example, we have

$${}^{1}\hat{x}=[\ 1-\sqrt[p]{(0.6)^p-(0.11)^p}\quad 1\quad 1\quad 0.4\ ],$$

${}^{2}\hat{x}=[1\quad 0.63\quad 0.63\quad 1]$, and then the maximum solution is

$$\hat{x}=[\ 1-\sqrt[p]{(0.6)^p-(0.11)^p}\quad 0.63\quad 0.63\quad 0.4\ ]$$

. Also,

$$M = \begin{bmatrix} 1-\sqrt[p]{(0.6)^p-(0.5)^p} & 1 & 0.4 & \infty \\ 1 & 0.62 & 1 & 1 \\ \infty & \infty & 0.32 & \infty \\ \infty & 0.6 & 0.6 & 1-\sqrt[p]{(0.4)^p-(0.2)^p} \end{bmatrix}$$

$$M^* = \begin{bmatrix} \infty & \infty & 0.4 & \infty \\ \infty & 0.62 & \infty & \infty \\ \infty & \infty & 0.32 & \infty \\ \infty & 0.6 & 0.6 & \infty \end{bmatrix}$$

This example is feasible by part (c) of Theorem 2. Also, two vectors ${}^1\check{x} = [0 \quad 0.62 \quad 0.6 \quad 0]$ and ${}^2\check{x} = [0 \quad 0.62 \quad 0.4 \quad 0]$ are generated. By pairwise comparison, the solution ${}^1\check{x}$ is removed and the solution ${}^2\check{x}$ is introduced as a single minimum solution. Therefore, $\check{x}(e^*) = {}^2\check{x}$. Moreover, $J' = \{1,2,3\}$, $J'' = \{4\}$, $x' = (x_1, x_2, x_3)$ and $x'' = x_4$. Also, $h_1 : [0,1]^3 \to [0,1]$ and $h_2 : [0,1] \to [0,1]$ where $h_1(x_1, x_2, x_3) = 0.5x_1 + 3x_2 + x_3$ is an increasing function and $h_2(x_4) = -x_4$ is a decreasing function. Finally, $f(x) = h_1(x') + h_2(x'')$ is a reducible function with monotone factors. Thus, Theorem 4 implies

$$x^* = [x_1^*, x_2^*, x_3^*, x_4^*] = [{}^2\check{x}_1, {}^2\check{x}_2, {}^2\check{x}_3, \hat{x}_4] = [0 \quad 0.62 \quad 0.4 \quad 0.4]$$

and $f(x^*) = 1.86$

Example (5): With the same assumptions as in Example 4, we define

$$f(x) = (1 \vee x_1) \wedge (\frac{1}{2} \vee x_2^2) \wedge (\frac{1}{4} \vee x_3^{-2}) \wedge (\frac{1}{12} \vee x_4^{-3})$$

where "$\wedge$" and "$\vee$" denote min operator and max operator, respectively. In this example, $J' = \{1,2\}$, $J'' = \{3,4\}$, $x' = (x_1, x_2)$ and $x'' = (x_3, x_4)$. Also, $h_1 : [0,1]^2 \to [0,1]$ and $h_2 : [0,1]^2 \to [0,1]$ where $h_1(x_1, x_2) = (1 \vee x_1) \wedge (\frac{1}{2} \vee x_2^2)$ is an increasing function and $h_2(x_3, x_4) = (\frac{1}{4} \vee x_3^{-2}) \wedge (\frac{1}{12} \vee x_4^{-3})$ is a decreasing function. Finally, $f(x) = h_1(x') \wedge h_2(x'')$ is a reducible function with monotone factors. Since ${}^2\check{x}$ is a single minimal solution (and then single minimum solution) we have $\check{x}(e^*) = {}^2\check{x}$. Therefore, by from Theorem 4 we attain

$$x^* = [x_1^*, x_2^*, x_3^*, x_4^*] = [{}^2\check{x}_1, {}^2\check{x}_2, \hat{x}_3, \hat{x}_4] = [0 \quad 0.62 \quad 0.63 \quad 0.4]$$

and $f(x^*) = 0.5$

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